

Essential Spectra of Linear Relations

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Abstract

Five essential spectra of linear relations are defined in terms of semi-Fredholm properties and the index. Basic properties of these sets are established and the perturbation theory for semi-Fredholm relations is then applied to verify a generalisation of Weyl's theorem for single-valued operators. We conclude with a Möbius transform spectral mapping theorem.

1 Introduction

While the study of the spectrum of bounded linear operators generalises the theory of eigenvalues of matrices, the essential spectra of linear operators characterise the non-invertibility of operators $\lambda - T$. The latter have been considered in terms of two key related directions of investigation, namely the study of the ascent and descent (as well as the nullity and defect) of $\lambda - T$ and in terms of semi-Fredholm properties of $\lambda - T$. Today there are several related definitions of essential spectra and comprehensive reviews may be found in [17], [22], [23], [24], [27] and [34]. In [19], the refinements of the spectrum in terms of ascent and descent were investigated in terms of states of operators, using the terminology of [31] (see also [8] for the states of linear relations). On the other hand, the perturbation theory of semi-Fredholm operators provides a more general context for the early observations of H. Weyl, who showed that limit points of the spectrum (i.e. all points of the spectrum, except isolated eigenvalues of finite multiplicity) of a bounded symmetric transformation on a Hilbert space are invariant under perturbation by compact symmetric operators (cf. Riesz and Sz-Nagy [28]).

In this paper we apply the theory of Fredholm relations to show that theory for essential spectra of linear operators can be extended naturally to linear relations. In particular, we extend preliminary results of Cross [8], where the set $\sigma_{e1}(-)$ defined below is introduced. The definitions in this paper are based on the classifications given in Edmunds and Evans [9] for single-valued operators.

We commence with a recollection of some preliminary properties required in the sequel.

2 Semi-Fredholm Linear Relations

We first clarify some notation and terminology. Let X and Y be normed linear spaces, and let $B(X, Y)$ and $L(X, Y)$ denote the classes of bounded and unbounded linear operators, respectively, from X into Y . A **multivalued linear operator** $T : X \rightarrow Y$ is a set-valued map such that its graph $G(T) = \{(x, y) \in X \times Y \mid y \in Tx\}$ is a linear subspace of $X \times Y$. We use the term **linear relation** or simply relation, to refer to such a multivalued linear operator denoted $T \in LR(X, Y)$ (cf. Arens [2] and Lee and Nashed [21]). A relation $T \in LR(X, Y)$ is said to be **closed** if its graph $G(T)$ is a closed subspace. The **closure** of a linear relation T , denoted \overline{T} is defined in terms of its corresponding graph: $G(\overline{T}) := \overline{G(T)} \subset X \times Y$.

The **conjugate** T' (cf [8], III.1.1) of a linear relation $T \in LR(X, Y)$ is defined by

$$G(T') := G(-T^{-1})^\perp \subset Y' \times X'$$

where $[(y, x), (y', x')] := [x, x'] + [y, y'] = x'x + y'y$. For $(y', x') \in G(T')$ we have $y'y = x'x$ whenever $x \in D(T)$.

Let Q_T , or simply Q , when there is no ambiguity about the relation T , denote the natural quotient map $Q_{\frac{Y}{T(0)}} : Y \rightarrow Y/\overline{T(0)}$ with kernel $\overline{T(0)}$. For $x \in D(T)$ define $\|Tx\|$ by

$$\|Tx\| := \|QTx\|,$$

and let the quantity $\|T\|$ be defined

$$\|T\| := \|QT\|.$$

Clearly QT is a single-valued linear operator. It follows from the definition that $\|Tx\| = d(y, T(0))$ for all $y \in Tx$, and that $\|T\| = \sup_{x \in B_{D(T)}} \|Tx\|$. The quantity $\|T\|$ is referred as the **norm** of T , though we note that it is in fact a pseudonorm since $\|T\| = 0$ does not imply $T = 0$.

A relation $T \in LR(X, Y)$ is said to be **continuous** if for any neighbourhood $V \subset R(T)$, the inverse image $T^{-1}(V) := \{u \in D(T) \mid V \cap Tu \neq \emptyset\}$ is a neighbourhood in $D(T)$, and T is said to be **open** if its inverse T^{-1} is continuous. It can be shown that T is continuous if and only if $\|T\| < \infty$ (cf. [8], II.3.2).

The **minimum modulus** of $T \in LR(X, Y)$ is the quantity

$$\gamma(T) := \sup \{ \lambda \in \mathbb{R} : \|Tx\| \geq \lambda d(x, N(T)) \text{ for } x \in D(T) \},$$

and T is *open* if and only if $\gamma(T) > 0$ ([8], II.3.2). The quantity $\gamma(T)$ is related to the norm quantity by $\gamma(T) = \|T^{-1}\|^{-1}$.

The **nullity** and **deficiency** of a linear relation $T \in LR(X, Y)$ are defined respectively as follows:

$$\begin{aligned} \alpha(T) &:= \dim N(T), \quad \text{and} \\ \beta(T) &:= \operatorname{codim} R(T) := \dim Y/R(T). \end{aligned}$$

If either $\alpha(T) < \infty$ or $\beta(T) < \infty$, then the **index** of T is defined as follows:

$$\kappa(T) := \alpha(T) - \beta(T),$$

where the value of the difference is computed as $\kappa(T) := \infty$ if $\alpha(T)$ is infinite and $\beta(T) < \infty$ and $\kappa(T) := -\infty$ if $\beta(T)$ is infinite and $\alpha(T) < \infty$.

If X and Y are Banach spaces and $T : X \rightarrow Y$ is a closed single-valued operator, then T is said to be a **Fredholm operator**, usually denoted $T \in \Phi(X, Y)$, if $R(T)$ is closed and both $\alpha(T) < \infty$ and $\beta(T) < \infty$; T is said to be **upper semi-Fredholm**, denoted $T \in \Phi_+(X, Y)$, if $R(T)$ is closed and $\alpha(T) < \infty$; and T is said to be **lower semi-Fredholm**, denoted $T \in \Phi_-(X, Y)$, if $R(T)$ is closed and $\beta(T) < \infty$.

Definitions 2.1. The **essential resolvent sets**, $\rho_{ei}(T)$ for $i = 1, 2, 3, 4, 5$, of $T \in LR(X)$ are defined as follows:

$$\begin{aligned} \rho_{e1}(T) &:= \{ \lambda \in \mathbb{C} \mid (\lambda - T) \in \Phi_+ \cup \Phi_- \} \\ \rho_{e2}(T) &:= \{ \lambda \in \mathbb{C} \mid (\lambda - T) \in \Phi_+ \} \\ \rho_{e3}(T) &:= \{ \lambda \in \mathbb{C} \mid (\lambda - T) \in \Phi \} \\ \rho_{e4}(T) &:= \{ \lambda \in \mathbb{C} \mid (\lambda - T) \in \Phi \text{ and } \kappa(\lambda - T) = 0 \} \\ \rho_{e5}(T) &:= \bigcup \rho_{e1}^{(n)}(T) \text{ where } \rho_{e1}^{(n)}(T) \text{ is a component of } \rho_{e1}(T) \\ &\quad \text{and } \rho_{e1}^{(n)}(T) \cap \rho(T) \neq \emptyset \end{aligned}$$

The **essential spectra**, $\sigma_{ei}(T)$, $i = 1, 2, 3, 4, 5$, of $T \in LR(X)$ are the respective complements of the essential resolvents:

$$\sigma_{ei}(T) := \mathbb{C} \setminus \rho_{ei}(T), \quad i = 1, 2, 3, 4, 5.$$

We also define

$$\begin{aligned}\rho'_{e2}(T) &:= \{ \lambda \in \mathbb{C} \mid (\lambda - T) \in \Phi_- \} \\ \sigma'_{e2}(T) &:= \mathbb{C} \setminus \rho'_{e2}(T)\end{aligned}$$

Clearly we have that $\rho_{ei}(T) \supset \rho_{ej}(T)$ for $i < j < 4$, and, thus, $\sigma_{ei}(T) \subset \sigma_{ej}(T)$ for $i < j < 4$. We will see later that $\rho_{e4}(T) \supset \rho_{e5}(T)$.

For the rest of this section we recall a selection of results from Cross [8] which are used in the sequel .

Proposition 2.2. *If $T \in LR(X, Y)$ is continuous with finite dimensional range, then T is compact.*

Proposition 2.3. *The following are equivalent:*

- (i) $T \notin \Phi_+$.
- (ii) *There exists a non-precompact bounded subset W of $D(T)$.*
- (iii) *T has a singular sequence.*

Proposition 2.4. *Let $T \in LR(X, Y)$ with $\gamma(T) > 0$. Suppose $S \in LR(X, Y)$ satisfies $D(S) \supset D(T)$, $S(0) \subset \overline{T(0)}$ and $\|S\| < \gamma(T)$. Then $\alpha(T + S) \leq \alpha(T)$ and $\bar{\beta}(T + S) \leq \bar{\beta}(T)$.*

The next result is a general version of the so-called small perturbation theorem for linear relations.

Proposition 2.5. *Let $S, T \in LR(X, Y)$. If $S(0) \subset \overline{T(0)}$ then $\Delta(S) < \Gamma(T) \Rightarrow T + S \in \Phi_+$, where*

$$\Gamma(T) := \inf_{M \in \mathcal{I}(\mathcal{D}(T))} \|T|_M\|, \quad \Delta(S) := \sup_{M \in \mathcal{I}(\mathcal{D}(S))} \Gamma(S|_M),$$

and $\mathcal{I}(\mathcal{X})$ denotes the collection of infinite dimensional subsets of X .

Proposition 2.6. *Let $T \in \Phi(X, Y)$ and suppose $S \in LR(X, Y)$ satisfies $D(S) \supset D(T)$, $S(0) \subset \overline{T(0)}$ and $\|S\| < \gamma(T)$, then $\kappa(T + S) = \kappa(T)$.*

Proposition 2.7. *Let $S, T \in LR(X, Y)$, $D(S) \supset D(T)$ and let $T \in \Phi_-$.*

- (a) *If $\dim R(S) < \infty$, then $T + S \in \Phi_-$.*
- (b) *If S is precompact, then $T + S \in \Phi_-$.*
- (c) *If $\|S\| < \gamma(T)$, then $T + S \in \Phi_-$.*

Proposition 2.8. (a) *Suppose $T \in \Phi_+(X, Y)$ and $S \in LR(X, Y)$ is strictly singular. If $\|S\| < \infty$, $D(S) \supset D(T)$, $S(0) \subset \overline{T(0)}$, then $\kappa(T + S) = \kappa(T)$.*

(b) *Suppose $T \in \Phi_-(X, Y)$ and $S \in LR(X, Y)$ is such that S' is strictly singular. If $\|S'\| < \infty$, $D(S) \supset D(T)$, $S(0) \subset \overline{T(0)}$, then $\kappa(T + S) = \kappa(T)$.*

3 Properties of the Essential Spectra

We begin this section by showing that the various essential spectra are closed, and then illustrate some characteristic properties. In the single-valued case, the set $\bigcap_{P \in \mathcal{K}_T} \sigma(T + K)$ is referred to as the *Weyl essential spectrum*.

Proposition 3.4 shows that $\sigma_{e4}(T)$ can be characterised in terms of the Weyl essential spectrum in the multivalued case as well (cf. Edmunds and Evans [9]). We conclude this section by giving properties of the quantities $\alpha(\lambda - T)$, $\beta(\lambda - T)$ and $\kappa(\lambda - T)$ for λ in the essential spectra, and deduce in Proposition 3.9 the inclusions

$$\sigma_{e1}(T) \subset \sigma_{e2}(T) \subset \sigma_{e3}(T) \subset \sigma_{e4}(T) \subset \sigma_{e5}(T) \subset \sigma(T).$$

Proposition 3.5 is included here for application in Proposition 3.9 and is based on the single-valued analogue given in Goldberg [13].

Proposition 3.1. *For $i = 1, 2, 3, 4, 5$, $\sigma_{ei}(T)$ is closed.*

PROOF

Suppose $\lambda \in \rho_{ei}(T)$, $i = 1, 2, 3, 4, 5$. Since $R(\lambda - T)$ is closed, it follows from the Open Mapping Theorem ([8], III.4.2), that $\gamma(\lambda - T) > 0$. If $\lambda - T \in \mathcal{F}_+$ and $|\mu| < \gamma(\lambda - T)$, then by Theorem 2.5, $\mu + \lambda - T \in \mathcal{F}_+$. Similarly, if $\lambda - T \in \mathcal{F}_-$ and $|\mu| < \gamma(\lambda - T)$, then by Theorem 2.7, $\mu + \lambda - T \in \mathcal{F}_-$. Thus, $\rho_{e1}(T)$, $\rho_{e2}(T)$ and $\rho_{e3}(T)$ are open. Furthermore, by Theorem 2.6, $\kappa(\mu + \lambda - T) = \kappa(\lambda - T)$, i.e. $\rho_{e4}(T)$ is open. Since each component of $\rho_{e1}(T)$ is open, so is $\rho_{e5}(T)$.

Proposition 3.2. *Let $T \in LR(X)$. Then*

- (a) $\sigma_{ei}(T') = \sigma_{ei}(T)$ for $i = 1, 3, 4, 5$
- (b) $\sigma_{e2}(T') = \sigma'_{e2}(T)$

PROOF

(a) Suppose $\lambda \in \rho_{ei}(T)$, $i = 1, 3, 4$. By [8], III.7.2, $\alpha(\lambda - T') = \beta(\lambda - T)$ since $R(\lambda - T)$ is closed. By the Closed Range Theorem ([8], III.4.4), $R(\lambda - T')$ if and only if $R(\lambda - T)$ is closed and, since $\lambda - T$ is open, $\beta(\lambda - T') = \alpha(\lambda - T)$. Thus, the result holds for $i = 1, 3$ and 4. Since $\rho_{e1}(T) = \rho_{e1}(T')$ and $\rho(T) = \rho(T')$, it follows that $\rho_{e1}^{(n)}(T') = \rho_{e1}^{(n)}(T)$, i.e. the result holds for $i = 5$.

(b) follows from the reasons given in (a).

Proposition 3.3. $\lambda \in \sigma_{e2}(T)$ if and only if $\lambda - T$ has a singular sequence.

PROOF

Since $\lambda \in \sigma_{e2}(T)$ if and only if $\lambda - T \notin \mathcal{F}_+$, the result follows from Theorem 2.3.

Proposition 3.4.

$$\sigma_{e4}(T) = \bigcap_{K \in \mathcal{K}_T} \sigma(T + K),$$

where $\mathcal{K}_T := \{K \in LR(X) \mid K \text{ is compact and } K(0) \subset \overline{T(0)}\}$.

PROOF

We show first that $\sigma_{e4}(T) \subset \bigcap_{K \in \mathcal{K}_T} \sigma(T + K)$. Suppose $\lambda \notin \bigcap_{K \in \mathcal{K}_T} \sigma(T + K)$. Then there exists $K \in \mathcal{K}_T$ such that $\lambda \in \rho(T + K)$. Thus $\lambda \in \rho_{e4}(T + K)$. By Propositions 2.5 and 2.7, $\lambda - T = \lambda - T - K + K \in \Phi$, and by Theorem 2.8,

$$\kappa(\lambda - T) = \kappa(\lambda - T - K + K) = \kappa(\lambda - T - K).$$

Thus, $\lambda \in \rho_{e4}(T)$, i.e. $\lambda \notin \sigma_{e4}(T)$.

Conversely, suppose $\lambda \in \rho_{e4}(T)$. Then $R(\lambda - T)$ is closed, and $\alpha(\lambda - T) = \beta(\lambda - T) = n$, say. Let $\{x_1, \dots, x_n\}$ and $\{y'_1, \dots, y'_n\}$ be bases for $N(\lambda - T)$ and $R(\lambda - T)^\perp = N(\lambda - T')$, respectively. Choose $x'_j \in X'$ and $y_j \in X$, $j = 1, \dots, n$ such that

$$\begin{aligned} x'_j x_k &= \delta_{jk}, \text{ and} \\ y'_j y_k &= \delta_{jk}, \end{aligned}$$

where $\delta_{jk} = 0$ if $j \neq k$ and $\delta_{jk} = 1$ if $j = k$, and define $K \in LR(X)$ as follows:

$$Kx := \sum_{k=1}^n (x'_k x) y_k, \quad x \in X$$

Then $\dim R(K) < \infty$ and

$$\|Kx\| \leq \left(\sum_{k=1}^n \|x'_k\| \|y_k\| \right) \|x\|.$$

By Proposition 2.2, it follows that K is a compact operator. By Propositions 2.5 and 2.7, it follows that $\lambda - (T + K) \in \Phi$ and by Theorem 2.8, $\kappa(\lambda - (T + K)) = \kappa(\lambda - T)$.

Without loss of generality, assume $\lambda = 0$. Now if $x \in N(T)$, then $x = \sum_{k=1}^n a_k x_k$ and $x'_j(x) = a_j$, $1 \leq j \leq n$. On the other hand, if $x \in N(K)$, then $x'_j(x) = 0$. Thus $N(T) \cap N(K) = 0$.

Similarly, if $y \in R(K)$, then $y = \sum_{k=1}^n a_k y_k$ and $y'_j(y) = a_j$, $1 \leq j \leq n$, and if $y \in R(T)$, then $y'_j(y) = 0$. Thus $R(K) \cap R(T) = 0$.

Next, suppose $x \in N(T + K)$. Then $Tx = -Kx + T(0)$. It follows from the argument above, that $Tx = T(0)$, i.e. $x \in N(T)$. Thus, $x = \sum_{k=1}^n a_k x_k$ and $x'_k(x) = a_k$, $1 \leq k \leq n$. Since $Kx = \sum_{k=1}^n (x'_k x) y_k = 0$, it follows that $x'_k(x) = 0$, $1 \leq k \leq n$, and hence $x = 0$. Thus, $\alpha(T + K) = 0 = \beta(T + K)$, i.e. $0 \in \rho_{e4}(T + K)$.

Proposition 3.5. *Suppose $T \in \Phi_+ \cup \Phi_-$ and $S \in LR(X, Y)$ satisfies $D(S) \supset D(T)$, $S(0) = \overline{S(0)} \subset \overline{T(0)}$, and $\|S\| < \gamma(T)$. Then $\exists \nu > 0$ such that $\alpha(T + \lambda S)$ and $\beta(T + \lambda S)$ are constant in the annulus $0 < |\lambda| < \nu$.*

PROOF

We first assume $\alpha(T) < \infty$. Let $\lambda \neq 0$ and let $x \in N(T + \lambda S)$. Then

$$Tx \supset -\lambda Sx,$$

whence

$$\begin{aligned} Sx &\subset R(T) =: R_1, \text{ and} \\ x &\in S^{-1}R_1 =: D_1. \end{aligned}$$

Thus

$$\begin{aligned} -\lambda Sx &\subset Tx \subset TD_1 =: R_2, \text{ and} \\ x &\in S^{-1}R_2 =: D_2. \end{aligned}$$

Proceeding in this way, we obtain

$$R_{k+1} := TD_k, \text{ where } D_k := S^{-1}R_k.$$

Clearly

$$R_1 \supset R_2 \supset \dots \text{ and } D_1 \supset D_2 \supset \dots$$

It follows from the construction of these sequences of subspaces that

$$N(T + \lambda S) \subset \bigcap_{k=1}^{\infty} D_k. \quad (1)$$

By induction, we have that R_n are closed subspaces of Y , and D_n are relatively closed subspaces of $D(S)$: from the hypothesis, R_1 is closed, and, hence, since S is continuous, and $S(0)$ is closed, D_1 is relatively closed in $D(S)$; if R_k and D_k are closed and relatively closed, respectively, then, since $T|_{D_k} \in \Phi_+ \cup \Phi_-$, it follows that $R_{k+1} = TD_k$ is closed, and, since S is continuous, and $S(0)$ is closed, $D_{k+1} = S^{-1}R_{k+1}$ is relatively closed in $D(S)$.

Define

$$\begin{aligned} X_1 &:= \bigcap_{k=1}^{\infty} D_k, \text{ and} \\ Y_1 &:= \bigcap_{k=1}^{\infty} R_k. \end{aligned}$$

Then, by the definitions of R_k and D_k , it follows that

$$TX_1 \subset Y_1 \text{ and } SX_1 \subset Y_1.$$

Now define T_1 and S_1 by :

$$T_1 := T|_{D(T) \cap X_1}, \text{ and } S_1 := S|_{D(T) \cap X_1}.$$

Then $R(T_1) \subset Y_1$ and $R(S_1) \subset Y_1$, and since T is closed and X_1 is relatively closed in $D(S)$ and hence also in $D(T)$, T_1 is a closed relation. To see that T_1 is surjective, let $y \in Y_1 = \bigcap_{n=1}^{\infty} TD_n$. Then for each $n \geq 1$, there exists $x_n \in D_n$ such that $y \in Tx_n$. Since $\alpha(T) < \infty$ and $D_n \supset D_{n+1}$, there exists k_0 such that for $k \geq k_0$,

$$N(T) \cap D_{k_0} = N(T) \cap D_k,$$

and for $x_k \in D_k$, and $x_{k_0} \in D_{k_0}$,

$$x_k - x_{k_0} \in N(T) \cap D_{k_0} = N(T) \cap D_k \subset D_k.$$

From this it follows that

$$x_{k_0} \in \bigcap_{k \geq k_0} D_k = X_1, \text{ and } y \in Tx_{k_0}.$$

i.e. T_1 is surjective. By the Open Mapping Theorem ([8], III.4.2), T_1 is open.

By Theorem 2.4, Propositions 2.5 and 2.7, and by Theorem 2.6, $\exists \nu > 0$ such that for $|\lambda| < \nu$ we have

$$\text{Since } \kappa(T + \lambda S) = \kappa(T). \quad (2)$$

$$\beta(T_1 + \lambda S_1) \leq \beta(T_1) = \bar{\beta}(T_1) = 0, \quad (3)$$

it follows that $\beta(T_1 + \lambda S_1) = 0$, and hence

$$\alpha(T_1 + \lambda S_1) = \kappa(T_1 + \lambda S_1) = \kappa(T_1) = \alpha(T_1). \quad (4)$$

By (1), it follows that for $\lambda \neq 0$,

$$N(T + \lambda S) = N(T_1 + \lambda S_1). \quad (5)$$

In particular, $\alpha(T + \lambda S) = \alpha(T_1 + \lambda S_1)$. By (2), (3), (4) and (5) it follows that $\alpha(T + \lambda S)$ and $\beta(T + \lambda S)$ are constant in the annulus $0 < |\lambda| < \nu$.

If $\alpha(T) = \infty$, then $\beta(T) < \infty$, and the result is obtained by passing to the conjugates.

Proposition 3.6. *If $\rho_{ei}^{(n)}(T)$ is a component of $\rho_{ei}(T)$, $i = 1, 2, 3$, then $\alpha(\lambda - T)$ and $\beta(\lambda - T)$ have constant values, n_1 and n_2 , respectively, $n_1, n_2 \in \mathbb{N} \cup \{\infty\}$, except perhaps at isolated points where*

$$\alpha(\lambda - T) > n_1 \text{ and } \beta(\lambda - T) > n_2.$$

PROOF

We first prove the result for the quantities $\alpha(\lambda - T)$. Since any component of an open set in \mathbb{C} is open, we have that $\rho_{ei}^{(n)}(T)$ are open sets. We first consider the case $\rho_{e1}^{(n)}(T)$. If $\alpha(\lambda - T) = \infty$ for all $\lambda \in \rho_{e1}^{(n)}(T)$, then we are done. Now suppose $\alpha(\lambda - T) < \infty$ for some $\lambda \in \rho_{e1}^{(n)}(T)$, define $\alpha(\lambda) := \alpha(\lambda - T)$, and choose λ_0 such that $\alpha(\lambda_0) = n_1$ is the smallest non-negative integer attained by $\alpha(\lambda)$ on $\rho_{e1}^{(n)}(T)$. Suppose $\alpha(\lambda') \neq n_1$ for some λ' . Since $\rho_{e1}^{(n)}(T)$ is connected, there exists an arc Λ in $\rho_{e1}^{(n)}(T)$ with endpoints λ_0 and λ' . Since $\lambda - T \in \Phi_+ \cup \Phi_-$, it follows from Proposition 3.5 that for each $\mu \in \Lambda$ there exists an open ball B_μ contained in $\rho_{e1}^{(n)}(T)$ such that $\alpha(\lambda)$ is constant on $B_\mu \setminus \{\mu\}$. Since Λ is compact, there exists a finite set of points $\lambda_1, \lambda_2, \dots, \lambda_n = \lambda'$ such that $B_{\lambda_0}, B_{\lambda_1}, \dots, B_{\lambda_n}$ cover Λ , and, for $0 \leq i \leq n-1$,

$$B_{\lambda_i} \cap B_{\lambda_{i+1}} \neq \emptyset. \quad (6)$$

It follows from Theorem 2.4 that $\alpha(\lambda) \leq \alpha(\lambda_0)$ for λ sufficiently close to λ_0 . Thus, since $\alpha(\lambda_0)$ is the minimum value attained by $\alpha(\lambda)$ on $\rho_{e1}^{(n)}(T)$, it follows that $\alpha(\lambda) = \alpha(\lambda_0)$ for λ sufficiently close to λ_0 . Since $\alpha(\lambda)$ is constant for all $\lambda \neq \lambda_0$ in B_{λ_0} , this constant must be $\alpha(\lambda_0)$. Similarly $\alpha(\lambda)$ is constant on $B_{\lambda_i} \setminus \{\lambda_i\}$ for $1 \leq i \leq n$. Thus, by (6) that $\alpha(\lambda) = \alpha(\lambda_0)$ for all $\lambda \in B_{\lambda'} \setminus \{\lambda'\}$ and $\alpha(\lambda') > n_1$.

To see that the result holds for $\beta(\lambda - T)$, we pass to the conjugate of T and apply the above, and the equality

$$\alpha(\lambda - T') = \beta(\lambda - T).$$

The proofs for $\rho_{e2}^{(n)}(T)$ and $\rho_{e3}^{(n)}(T)$ are similar.

Proposition 3.7. $\lambda \in \rho_{e5}(T)$ if and only if $\lambda \in \rho_{e4}(T)$ and a deleted neighbourhood of λ lies in $\rho(T)$.

PROOF

Suppose $\lambda \in \rho_{e5}(T)$. Then, by definition, λ lies in a component $\rho_{e1}^{(n)}(T)$ of $\rho_{e1}(T)$ which intersects $\rho(T)$. Let C be such a component. Clearly $C \cap \rho(T)$ is open.

Since $\mu \in C \cap \rho(T)$ implies $\alpha(\mu - T) = \beta(\mu - T) = \kappa(\mu - T) = 0$, it follows from Theorem 2.6 that $\kappa(\lambda - T) = 0$ for $\lambda \in C$ when λ is sufficiently close to μ , and, hence for all $\lambda \in C$. Applying Proposition 3.6, we see that $\alpha(\lambda - T) = \beta(\lambda - T) = 0$ for all except some isolated points, say λ_j where $\alpha(\lambda_j - T) > 0$ and $\beta(\lambda_j - T) > 0$. Thus if $\lambda \in \rho_{e5}(T)$, then either $\lambda \in \rho(T)$ or λ is one of these isolated points in $\rho_{e4}(T)$.

Clearly the converse is true.

Corollary 3.8. If $\rho_{e4}(T)$ is connected and $\rho(T) \neq \emptyset$, then $\rho_{e5}(T) = \rho_{e4}(T)$.

PROOF

Since $\rho(T) \subset \rho_{e4}(T)$, it follows from the hypothesis and Proposition 3.6 that $\alpha(\lambda - T) = \beta(\lambda - T) = 0$ for all $\lambda \in \rho_{e4}(T)$ except perhaps at isolated points, i.e. a deleted neighbourhood of λ lies in $\rho(T)$. The result follows from Proposition 3.7.

Proposition 3.9.

$$\sigma_{e1}(T) \subset \sigma_{e2}(T) \subset \sigma_{e3}(T) \subset \sigma_{e4}(T) \subset \sigma_{e5}(T) \subset \sigma(T)$$

PROOF

Clearly

$$\rho_{e1}(T) \supset \rho_{e2}(T) \supset \rho_{e3}(T) \supset \rho_{e4}(T).$$

The remaining inclusions follow from Proposition 3.7.

Proposition 3.10. The index is constant in each connected component $\rho_{ek}^{(n)}(T)$ of $\rho_{ek}(T)$, $k = 1, 2, 3, 4, 5$.

PROOF

Clearly the result holds for $\rho_{e4}^{(n)}(T)$, and it follows from Proposition 3.7 that the result hold for $\rho_{e5}^{(n)}(T)$.

Let λ and λ' be distinct points in $\rho_{ek}^{(n)}(T)$, $k = 1, 2, 3$. Let Λ be an arc in $\rho_{ek}^{(n)}(T)$ with endpoints λ and λ' . By Theorem 2.6, there exists $\epsilon > 0$ such that $\kappa(\mu - T) = \kappa(\lambda - T)$ for any μ such that $|\mu - \lambda| < \epsilon$. Clearly the open balls $B(\lambda)$, $\lambda \in \Lambda$ cover Λ . Since Λ is compact, a finite number of these balls suffices to cover Λ . Since each of these balls overlap, it follows that $\kappa(\lambda - T) = \kappa(\lambda' - T)$.

4 Perturbation of the Essential Spectra

We now apply perturbation theorems for semi-Fredholm relations to verify the stability properties of the essential spectra under small and compact perturbation. In particular we arrive at generalisations of Weyl's theorem for linear operators to a relatively compact case [4]. First we recall Propositions 4.1 to 4.3 which are proved in Cross [8].

Proposition 4.1. Let $T \in LR(X, Y)$ and let $G = G_T$ denote the graph operator of T , i.e. G_T is the identity injection of X_T into X ($G_T x = x$) and X_T is the vector space $D(T)$ endowed with the norm $\|x\|_T := \|x\| + \|Tx\|$ for $x \in D(T)$. Then TG is open if and only if T is open and

$$\gamma(TG) = \frac{\gamma(T)}{1 + \gamma(T)}, \text{ provided } T \neq 0,$$

with the cases $\frac{\infty}{\infty} := 1$ and $\gamma(TG) := \infty$ if $T = 0$.

Proposition 4.2. *The norms $\|\cdot\|_T$ and $\|\cdot\|_{\lambda-T}$ are equivalent.*

Proposition 4.3. *Let $T \in LR(X, Y)$ and suppose $S \in LR(X, Y)$ satisfies $D(S) \supset \overline{D(T)}$ and $S(0) \subset T(0)$, and is T -bounded with $a, b > 0$, $b < 1$ such that for $x \in D(T)$, $\|Sx\| \leq a\|x\| + b\|Tx\|$.*

(a) *The norms $\|\cdot\|_T$ and $\|\cdot\|_{T+S}$ are equivalent.*

(b) *If X and Y are complete and T is closed, then $T + S$ is closed.*

Theorem 4.4. *Let $T \in LR(X)$ be closed and suppose $S \in LR(X)$ is T -compact with T -bound $b < 1$ [4], and $D(S) \supset \overline{D(T)}$ and $S(0) \subset T(0)$. Then for $i = 1, 2, 3, 4$*

$$\sigma_{ei}(T + S) = \sigma_{ei}(T).$$

If additionally ρ_{e4} is connected and neither $\rho(T)$ nor $\rho(T + S)$ are empty, then

$$\sigma_{e5}(T + S) = \sigma_{e5}(T).$$

PROOF

By Corollary 4.2, the norms $\|\cdot\|_T$ and $\|\cdot\|_{\lambda-T}$ are equivalent and hence, S is $(\lambda - T)$ -compact. Let $G_{\lambda-T}$ denote the graph operator from space $X_{\lambda-T} := (X, \|\cdot\|_{\lambda-T})$ into X . Suppose $\lambda - T \in \Phi_{\pm}$. Clearly $R(TG_{\lambda-T}) = R(T)$, and as subsets of the set X , we have $N(TG_{\lambda-T}) = N(T)$. By Proposition 4.1, $(\lambda - T)G_{\lambda-T}$ is open, and hence $(\lambda - T)G_{\lambda-T} \in \Phi_{\pm}$. Thus, by Propositions 2.5 and 2.7, it follows that $(\lambda - T) - S = \lambda - (T + S) \in \Phi_{\pm}$ and by Theorem 2.8, $\kappa(\lambda - (T + S)) = \kappa(\lambda - T)$.

On the other hand, suppose $\lambda - (T + S) \in \Phi_{\pm}$. By the equivalence of the norms $\|\cdot\|_T$ and $\|\cdot\|_{\lambda-(T+S)}$ (Proposition 4.3 and Corollary 4.2), it follows that S is $(\lambda - (T + S))$ -compact. Arguing as before, it follows that $\lambda - T \in \Phi_{\pm}$ and $\kappa(\lambda - T) = \kappa(\lambda - (T + S))$.

Thus, $\rho_{ei}(T + S) = \rho_{ei}(T)$ for $i = 1, 2, 3, 4$. It follows from the additional hypotheses, Corollary 3.8, and what has just been proved that

$$\rho_{e5}(T) = \rho_{e4}(T) = \rho_{e4}(T + S) = \rho_{e5}(T + S).$$

5 Functions of the Essential Spectra

The Möbius transform, $\eta(\lambda) = (\mu - \lambda)^{-1}$, is a topological homeomorphism from $\mathbb{C} \cup \{\infty\}$, endowed with the usual topology, onto itself. Theorem 5.2 below is analogous to the Theorem on the Möbius transform of the spectrum in Cross [8]. For its proof, we first recall the following index theorem:

Proposition 5.1. *Let $T \in LR(X, Y)$ and $S \in LR(Y, Z)$. Suppose $D(S) = Y$ and that T and S have finite indices. Then*

$$\kappa(ST) = \kappa(T) + \kappa(S) - \dim(T(0) \cap N(S)).$$

Theorem 5.2. *Let $T \in LR(X)$ be closed. Suppose $\mu \in \rho(T)$. Then for $i = 1, 2, 3, 4, 5$*

$$\lambda \in \sigma_{ei}(T) \Leftrightarrow (\mu - \lambda)^{-1} \in \sigma_{ei}(T_{\mu}).$$

PROOF

Let $S := (\mu - \lambda)((\mu - \lambda)^{-1} - T_{\mu})$. It can be shown that $\lambda - T = S(\mu - T)$ ([8], IV.4.2). Since T is closed, so is $\lambda - T$, and since $R(\mu - T) = X$ it follows that

$$R(\lambda - T) = R(S). \tag{7}$$

Since T_{μ} is single valued,

$$\alpha(\lambda - T) = \dim T_{\mu}S^{-1}(0) \leq \dim S^{-1}(0) = \alpha(S).$$

Thus, $S \in \Phi_{\pm}$ implies that $\lambda - T \in \Phi_{\pm}$, i.e. $(\mu - \lambda)^{-1} \in \rho_{ei}(T_{\mu})$ implies that $\lambda \in \rho_{ei}(T)$ for $i = 1, 2, 3$. Applying Proposition elementary algebra for linear relations ([8], I.4.2) we have

$$\begin{aligned}
(\mu - T)S &= (\mu - T)(\mu - \lambda)((\mu - \lambda)^{-1} - T_{\mu}) \\
&= (\mu - T) - (\mu - \lambda)(\mu - T)(\mu - T)^{-1} \\
&= (\mu - T) - (\mu - \lambda)(I + (\mu - T)(\mu - T)^{-1} - (\mu - T)(\mu - T)^{-1}) \\
&= \lambda - T + (\mu - \lambda)(TT^{-1} - TT^{-1}) \\
&= \lambda - T.
\end{aligned}$$

Thus, since $\kappa(\mu - T)$ and $\kappa(S)$ are finite and $D(S) = X$, it follows from Proposition 5.1 that

$$\kappa(\lambda - T) = \kappa(S) + \kappa(\mu - T) - \dim(S(0) \cap N(\mu - T)). \quad (8)$$

In particular, if $(\mu - \lambda)^{-1} \in \rho_{e4}(T_{\mu})$ then $\kappa(S) = 0$, and, since $\mu \in \rho(T)$, we have $\kappa(\mu - T) = 0 = \alpha(\mu - T)$. Thus $\kappa(\lambda - T) = 0$, i.e. $\lambda \in \rho_{e4}(T)$. Applying Proposition 3.7, it follows that the forward implication also holds for $i = 5$.

For the reverse implication, it follows from (7) that if $\lambda - T \in \Phi_{-}$, then $S \in \Phi_{-}$, i.e. $(\mu - \lambda)^{-1} - T_{\mu} \in \Phi_{-}$. Now suppose $\lambda - T \in \Phi_{+}$. Then there exists a finite codimensional subset M of $D(\lambda - T)$ such that $(\lambda - T)|_M$ is injective. As in [8] IV.4.2, it follows that $S|_M$ is injective, and hence $\alpha(S) < \infty$. Thus, $S \in \Phi_{+}$, and consequently $(\mu - \lambda)^{-1} - T_{\mu} \in \Phi_{+}$. We have

$$\lambda \in \rho_{ei}(T) \Rightarrow (\mu - \lambda)^{-1} \in \rho_{ei}(T_{\mu}) \text{ for } i = 1, 2, 3.$$

Now if $\lambda \in \rho_{e4}(T)$ then $\kappa(\lambda - T) = 0$, and since $\alpha(\mu - T) = \kappa(\mu - T) = 0$ it follows from (8) that $0 = \kappa(S) = \kappa((\mu - \lambda)^{-1} - T_{\mu})$. Thus $(\mu - \lambda)^{-1} \in \rho_{e4}(T_{\mu})$. Another application of Proposition 3.7 shows that the converse is true for $i = 5$.

Theorem 5.3. *Let X be complete and let $T, S \in LR(X)$ be closed. Suppose $\mu \in \rho(T) \cap \rho(S)$ and $T_{\mu} - S_{\mu}$ is compact. Then for $i = 1, 2, 3, 4$*

$$\sigma_{ei}(S) = \sigma_{ei}(T).$$

If additionally $\rho_{e4}(S)$ is connected then equality holds for $i = 5$ as well.

PROOF

For $i = 1, 2, 3, 4$ it follows from Theorem 5.2, that

$$\lambda \in \sigma_{ei}(T) \Leftrightarrow (\lambda - \mu)^{-1} \in \sigma_{ei}(T_{\mu}),$$

and

$$\lambda \in \sigma_{ei}(S) \Leftrightarrow (\lambda - \mu)^{-1} \in \sigma_{ei}(S_{\mu}),$$

and by Theorem 4.4,

$$\sigma_{ei}(T_{\mu} - (T_{\mu} - S_{\mu})) = \sigma_{ei}(T_{\mu}).$$

Applying Proposition 3.7 shows that the result is true for $i = 5$ under the additional hypotheses.

6 Further Notes and Remarks

We note that Proposition 3.1 appeared for case σ_{e1} in [8] (VII.2.3) and that a similar but different generalisation of Weyl's theorem is proved in a lengthier argument through Theorems VII.2.15 and VII.2.3 of [8].

Other subsets of the spectrum of a linear operator have also been investigated for stability under perturbation, for example the *Browder essential spectrum* defined by :

$$\sigma_b(T) := \bigcup \{ \sigma(T + K) \mid TK = KT \text{ and } K \text{ is compact} \}.$$

It is possible that such investigations may be extended to multivalued linear operators by the methods employed in this work. More recently Sandovici, De Snoo and Winkler [29] have developed results for the ascent, descent, nullity and defect of linear relations.

For simplicity, we have assumed that the spaces on which the relations are defined are complete, and that the operators are closed. Fredholm properties are, however, stable under more general conditions (cf. Cross [8] for the case σ_{e1}). Thus, proofs for σ_{ei} , $i = 1, 2, 3$ do not necessarily require assumptions of completeness. The index may not be stable under perturbation, though, and hence, generalisations which weaken assumptions of completeness for σ_{ei} , $i = 4, 5$ would have to proceed with considerations similar to those applied for the class of Atkinson relations introduced in Wilcox [33] (see also L. Labuschagne [18] and V. Müller-Horrig [25]).

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References

- [1] T. Alvarez, R.W. Cross and D.L. Wilcox, *Perturbation theory of multivalued Atkinson operators in normed spaces*, Bull.Austral.Math.Soc. Vol. 76, 195-204 (2007)
- [2] R. Arens, *Operational Calculus of linear relations*, Pacific J. Math., 11, 9-23 (1961)
- [3] F.V. Atkinson, *On relatively regular operators*, Acta Sci. Mat Szeged 15, 38-56 (1953) [Russian]
- [4] P. Binding and R.Hryniv, *Relative boundedness and relative compactness for linear operators in Banach spaces* Proceedings of the American Mathematical Society, Vol 128, Number 8, Pages 2287-2290
- [5] F.E. Browder, *On the spectral theory of elliptic differential operators I.*, Math. Ann. 142, 22-130 (1961)
- [6] S.L. Campbell (editor), *Recent applications of generalised inverses*, Research Notes in Math. 66, Pitman Advanced Publishing, London, 1982
- [7] E.A. Coddington, *Self-adjoint problems for nondensely defined ordinary differential operators and their eigenfunction expansions*, Adv. in Math., 15 , 1-40 (1975)
- [8] R.W. Cross, *Multivalued Linear Operators*, Marcel-Dekker, New York, 1998
- [9] D.E. Edmunds and W.D. Evans, *Spectral Theory and Differential Operators*, Clarendon, Oxford, Oxford, 1987
- [10] A. Favini and A. Yagi, *Degenerate differential equations in Banach spaces*, Marcel Dekker Inc., New York, 1999
- [11] I. Fredholm, *Sur une classe d'equations fonctionnelles*, Acta math., 27, 365-390 (1903)
- [12] I.C. Gohberg and M.C. Krein, *The basic propositions on defect numbers, root numbers and indices of linear operators*, Uspehi Mat. Nauk 12, 2 (74), 43-118 (1957) [Russian]
- [13] S. Goldberg, *Unbounded Linear Operators*, McGraw-Hill, New York, 1966
- [14] M. González and V.M. Onieva, *Atkinson Operators in Locally Convex Spaces*, Math.Z. 190, 505-517 (1985)
- [15] K. Gustafson and J. Weidmann, *On the Essential Spectrum*, J. Math. Anal. Appl., 25, 121-127 (1969)

- [16] T. Kato, *Perturbation theory for nullity, deficiency and other quantities of linear operators*, J.Analyse Math. 6, 273-322 (1958)
- [17] V. Kordula and V. Müller *On the axiomatic theory of spectrum*, Studia Math., 119 (2), 109-128 (1996)
- [18] L.E. Labuschagne, *The Perturbation of Relatively Open Operators with Reduced Index*, Math. Proc. Cambridge Phil. Soc., 112, 385-402 (1992)
- [19] D.C. Lay, *Spectral Analysis Using Ascent, Descent, Nullity and Defect*, Math.Ann. 184, 197-214 (1970)
- [20] D.C. Lay, *Spectral Properties of Generalized Inverses of Linear Operators*, SIAM J. Appl. Math. Vol. 29, No.1 , 103-109 (1975)
- [21] S.J. Lee and M.Z. Nashed, *Least-square solutions to multi-valued linear operator equations in Hilbert spaces*, J. Approx. Theory, 8, 380-391, (1983)
- [22] M. Mbekhta, *On the generalised resolvent in Banach spaces*, J. Math. Anal. Appl. 189, 362 - 377 (1995)
- [23] M. Mbekhta and V. Muller, *On the axiomatic theory of spectrum. II*, Studia Math. 119 (2) , 129-147 (1996)
- [24] M. Möller, *On the essential spectrum of a class of operators in Hilbert space*, Math. Nachr. 194, 1, 185-196 (1998)
- [25] V. Müller-Horrig, *Zur Theori der Semi-fredholm-Operatoren met stetig projitziertem Kern und Bild*, Math. Nachr.99, 185-197 (1980)
- [26] M.Z. Nashed (editor), *Generalized inverses and applications*, Academic Press, New York, 1976
- [27] V. Rakočević. *Apostol spectrum and generalisations: a brief survey* Ser.Math.Inform. 14, 79-108 (1999)
- [28] F. Riesz and B. Sz.-Nagy, *Functional Analysis* translated from 2nd French edition by L.F. Boron, Frederick Ungar Publishing Co., New York, 1955
- [29] A. Sandovici, H. de Snoo, H. Winkler, *Ascent, descent, nullity, defect and related notions for linear relations in linear spaces* , Linear Algebra and its Applications 423, 456-497 (2007)
- [30] M. Schechter, *On the essential spectrum of an arbitrary linear operator I.*, J. Math. Anal. Appl. 13, 205 - 215 (1966)
- [31] A.E. Taylor, *Theorems on Ascent, Descent, Nullity and Defect of Linear Operators*, Math. Annalen 163, 18-49 (1966)
- [32] H. Weyl, *Über beschränkte quadratische Formen deren Differenz volstetig ist*, Rend. Circ. Mat. Palermo, 27, 572-580 (1909)
- [33] D.L. Wilcox, *Multivalued Semi-Fredholm Operators in Normed Linear Spaces*, Ph.D. thesis, University of Cape Town, 2002
- [34] W. Zelazko, *An axiomatic approach to joint spectra I*, Studia Math., T. LXIV, 249-261 (1979)